Permutations and Combinations

Basic principle of counting:
Suppose two experiments are to be performed. Then, if the first experiment can result in \( m \) outcomes and if for each outcome of the first experiment there are \( n \) outcomes of the second experiment, then all together there are \( m \times n \) possible outcomes.

Examples:

Permutations:
How many different ordered arrangements of the letters \( A, B, C \) are possible? There are 6 permutations: \( ABC, ACB, BAC, BCA, CAB, CBA \). Or, using the basic principle of counting we can find the number of permutations as follows: \( 3 \times 2 \times 1 = 6 \).

In general \( n \) objects can be ordered in \( n \times (n - 1) \times (n - 2) \times \cdots \times 1 = n! \) ways. Each arrangement it is called a permutation.

Example: In how many ways can 4 math books, 3 chemistry books, and 2 history books can be ordered so that books of the subject are together?
Suppose \( k \) objects are to be selected and ordered from \( n \) objects \((k < n)\). Say, \( n = 4 \) \((A, B, C, D)\), and \( k = 3 \). Let’s list all the possible permutations:

\[
\begin{align*}
     ABC & \quad BCD & \quad CDA & \quad DAB \\
     ABD & \quad BCA & \quad CDB & \quad DAC \\
     ACB & \quad BDA & \quad CAB & \quad DBC \\
     ACD & \quad BDC & \quad CAD & \quad DBA \\
     ADB & \quad BAC & \quad CBD & \quad DCA \\
     ADC & \quad BAD & \quad CBA & \quad DCB
\end{align*}
\]

As we observe there are 24 permutations. Much easier, we can find the number of permutations using the basic principle of counting as follows: \(4 \times 3 \times 2 = 24\).

In general, the number of ways that \( k \) objects can be selected and ordered from \( n \) objects are:

\[
\frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)}{(n-k)!} = \frac{n!}{(n-k)!}.
\]

Example: In how many ways can we select and order 5 cards from the 52 cards?

\[
\frac{52!}{(52-5)!} = \frac{52!}{47!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{47!} = 311875200.
\]

**Combinations:**
Order does not count. For example \( ABC, ACB, BAC, CAB, CBA, BCA \) are 6 permutations, but in terms of combinations they count for only one. For every 6 permutations we have one combination. In our example, if we divide 24 by 6 we get the number of combinations: \( \frac{24}{6} = \frac{24}{3!} = 4 \). In general: In how many ways can we select \( k \) objects from \( n \) objects:

\[
\frac{n!}{(n-k)!k!} = \binom{n}{k}.
\]

A poker hand consists of 5 cards. How many poker hands are there?

\[
\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2598960.
\]

Sometimes we refer to \( \binom{n}{k} \) as the *binomial coefficient* because it appears in the binomial theorem:

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

For example,

\[
(x + y)^4 = \binom{4}{0} x^0 y^4 + \binom{4}{1} x^1 y^3 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^3 y^1 + \binom{4}{4} x^4 y^0 = y^4 + 4xy^3 + 6x^2 y^2 + 4x^3 y + x^4.
\]