Binomial option pricing model

Define:

- $S_0$ Stock price at $t = 0$
- $S_1$ Stock price at $t = 1$
- $E$ Exercise price of the call option
- $u$ $1 + \%$ change in stock price from $t = 0$ to $t = 1$ if stock price increases ($u > 1$)
  $$u = e^{\sigma \sqrt{t}}$$
- $d$ $1 - \%$ change in stock price from $t = 0$ to $t = 1$ if stock price decreases ($d < 1$)
  $$d = \frac{1}{u}$$
- $C$ The call price
- $\alpha$ The number of shares of stocks purchased per one call (hedge ratio)
- $C_u$ Price of call at $t = 1$ if stock price increases: $\max(S_1 - E, 0)$ or $\max(uS_0 - E, 0)$
- $C_d$ Price of call at $t = 1$ if stock price decreases: $\max(S_1 - E, 0)$ or $\max(dS_0 - E, 0)$
- $r$ Interest rate

<table>
<thead>
<tr>
<th>Action</th>
<th>Flows at $t = 0$</th>
<th>Flows at $t = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write 1 call</td>
<td>$C$</td>
<td>$-C_u$</td>
</tr>
<tr>
<td>Buy $\alpha$ shares of stock</td>
<td>$-\alpha S_0$</td>
<td>$\alpha u S_0$</td>
</tr>
</tbody>
</table>

One-step binomial tree
This will be a hedged portfolio at \( t = 1 \) if the payoffs at \( t = 1 \) are independent of the price of the stock at \( t = 1 \), i.e. \(-C_u + \alpha u S_0 = -C_d + \alpha d S_0\). Solve for \( \alpha \) to get \( \alpha = \frac{C_u - C_d}{u S_0 - d S_0} \). The payoff at \( t = 1 \) is \( \alpha d S_0 - C_d \), and since this is riskless payoff, the portfolio we constructed at \( t = 0 \) must have earned the risk free interest rate. Therefore, \((\alpha S_0 - C)(1 + r) = \alpha d S_0 - C_d \). We can solve for \( C \) to get:

\[
C = \frac{-\alpha d S_0 + C_d + (1 + r)\alpha S_0}{1 + r}
\]

Also \( \alpha = \frac{C_u - C_d}{u S_0 - d S_0} \).

Using both equations we get:

\[
C = \frac{C_u \frac{1+r-d}{u-d} + C_d \frac{u-(1+r)}{u-d}}{1 + r}
\]

Now, let \( p = \frac{1+r-d}{u-d} \) and \( 1 - p = \frac{u-(1+r)}{u-d} \) to get

\[
C = \frac{C_u p + C_d (1 - p)}{1 + r}
\]  

This is the price of a European call with one period to expiration.

**Numerical example**

Find the price of a European call if \( S_0 = 50 \), \( E = 50 \), \( u = 1.05 \), \( d = 0.95 \), \( r = 1\% \) using one-step binomial tree. Find the price of a European put written on the same stock.
Two-step binomial tree

The previous result was for one period. The time to expiration is divided into many small intervals and the procedure above is iterated. We will examine the result for a two-step binomial tree and then generalize.

Let’s examine the upper branch (top circle) of this two-step binomial tree:

<table>
<thead>
<tr>
<th>Action</th>
<th>Flows at $t = 1$</th>
<th>Flows at $t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write 1 call</td>
<td>$C_u$</td>
<td>$-C_{u^2}$</td>
</tr>
<tr>
<td>Buy $\alpha$ shares of stock</td>
<td>$-\alpha u S_0$</td>
<td>$\alpha u^2 S_0$</td>
</tr>
</tbody>
</table>

This will be a hedged portfolio if $-C_{u^2} + \alpha u^2 S_0 = -C_{ud} + \alpha ud S_0$, and solving for $\alpha$ we get $\alpha = \frac{C_{u^2} - C_{ud}}{\alpha u S_0 - ud S_0}$. Since at $t = 2$ the payoff is riskless the portfolio that we constructed at time $t = 1$ must have earned the risk free interest rate, i.e. $(C_u + \alpha u S_0)(1 + r) = \alpha ud S_0 - C_{ud}$. Solve for $C_u$ to get:

$$C_u = \frac{C_{u^2}p + C_{ud}(1 - p)}{1 + r}.$$  

(2)

Similarly, we examine the lower branch (bottom circle) of the two-step binomial tree:

<table>
<thead>
<tr>
<th>Action</th>
<th>Flows at $t = 1$</th>
<th>Flows at $t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write 1 call</td>
<td>$C_d$</td>
<td>$-C_{ud}$</td>
</tr>
<tr>
<td>Buy $\alpha$ shares of stock</td>
<td>$-\alpha d S_0$</td>
<td>$\alpha d^2 S_0$</td>
</tr>
</tbody>
</table>

This will be a hedged portfolio if $-C_{ud} + \alpha ud S_0 = -C_{d^2} + \alpha d^2 S_0$, and solving for $\alpha$ we get $\alpha = \frac{C_{ud} - C_{d^2}}{\alpha d S_0 - d^2 S_0}$. Since at $t = 2$ the payoff is riskless the portfolio that we constructed at time $t = 1$ must have earned the risk free interest rate, i.e. $(\alpha d S_0 - C_u)(1 + r) = -C_{d^2} + \alpha d^2 S_0$. Solve for $C_d$ to get:

$$C_d = \frac{C_{uap} + C_{d^2}(1 - p)}{1 + r}.$$  

(3)
Using equations (2) and (3) we update equation (1):

\[
C = \frac{C_u^2 p + C_u d (1 - p)}{1 + r} p + \frac{C_u d + C_d^2 (1 - p)}{1 + r} (1 - p)
\]

or

\[
C = \frac{C_u^2 p^2 + 2 C_u d p (1 - p) + C_d^2 (1 - p)^2}{(1 + r)^2}.
\]

This is the price of the call with two periods to expiration.

**Exercise**

Complete the three-step binomial tree and express \(C\) as a function of \(C_u^3, C_d^3, C_u d^2, C_u d^2, p, (1 - p), r\).
In general if we divide the time to expiration into \( n \) periods we get

\[
C = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} C_{u^j d^{n-j}}, \quad \text{where} \quad C_{u^j d^{n-j}} = \max(u^j d^{n-j} S_0 - E, 0).
\]

This expression can be simplified because the call is not always in the money at the end of the \( n \)th period. For the call to be in the money we want a minimum of \( k \) up movements of the stock. Therefore if we find \( k \) the summation will simply begin from \( j = k \). The call is in the money as long as \( u^k d^{n-k} S_0 - E > 0 \) and solving for \( k \) we get

\[
k = \frac{\log(\frac{u^k d^{n-k}}{E})}{\log(\frac{u}{d})}.
\]

And the price of the call is equal to:

\[
C = \sum_{j=k}^{n} \binom{n}{j} p^j (1 - p)^{n-j} C_{u^j d^{n-j}} = \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1 - p)^{n-j} C_{u^j d^{n-j}}
\]

\[
C = \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1 - p)^{n-j} \left( (u^j d^{n-j} S_0 - E) \right)
\]

\[
C = S_0 \left[ \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} (1 - p)^{n-j} \right] - \frac{E}{(1+r)^n} \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1 - p)^{n-j}
\]

Aside: If we let \( p' = \frac{pu}{1+r} \), then \( \frac{(pu)'^j [(1-p')d^{n-j}]}{(1+r)^n} = p'(1 - p')^{n-j} \) and the price of the call is equal to:

\[
C = S_0 \left[ \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p'^j (1 - p')^{n-j} \right] - \frac{E}{(1+r)^n} \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p'^j (1 - p')^{n-j}
\]

\[
C = S_0 P(X \geq k) - \frac{E}{(1+r)^n} P(Y \geq k), \quad \text{where} \quad X \sim b(n, p'), \quad \text{and} \quad Y \sim b(n, p).
\]

**Numerical example**

Using the binomial option pricing model find the price of a European call if \( S_0 = \$30, E = \$29, \sigma = 0.30, r = 0.05 \), with 73 days to expiration (\( \frac{73}{365} \) of a year), and \( n = 5 \) periods (five-step binomial tree).

\[
u = e^{\sigma \sqrt{\frac{t}{N}}} = exp(0.30 \sqrt{\frac{0.2}{5}}) = 1.061837.
\]

\[
d = e^{-\sigma \sqrt{\frac{t}{N}}} = \frac{1}{u} = 0.941764.
\]

\[
r_p = (1.05)^{\frac{1}{53}} - 1 = 0.001954.
\]

\[
p = \frac{1 + r_p - d}{u - d} = 0.50128.
\]

\[
p' = \frac{pu}{1 + r_p} = 0.53124.
\]

\[
k = \frac{\log\left(\frac{E}{X_1 X_0}\right)}{\log\left(\frac{u}{d}\right)} = 2.22 \Rightarrow k = 3.
\]

\[
C = 30 P(X \geq 3) - \frac{29}{1 + 0.001954^5} P(Y \geq 3) = 2.325.
\]

Note: \( X \sim b(5, 0.53124) \) and \( Y \sim b(5, 0.50128) \).

The R command:

\[
30*pbnotm(2, 5, 0.53124, \text{lower.tail}=\text{FALSE}) -
(29/(1+.001954^5))*pbnotm(2, 5, 0.50128, \text{lower.tail}=\text{FALSE})
\]

See next page for the complete five-step binomial tree.
$S_0 = 30$, $E = 29$, $\sigma = 0.30$

$\nu = 0.05$ per year

Time until expiration = 73 days (1/3 year)

$n = 5$ intervals

\[ C = ? \]

\[ \text{MAX} \left( S_t - E \right) \]

\[ 11.496 \text{ IN} \]

\[ 6.917 \text{ IN} \]

\[ 2.855 \text{ IN} \]

\[ 0 \text{ OUT} \]

\[ 0 \text{ OUT} \]

\[ 0 \text{ OUT} \]