Implied volatilities

One of the most important uses of the Black-Scholes model is the calculation of implied volatilities. These are the volatilities implied by the option prices observed in the market. Given the price of a call option, the implied volatility can be computed from the Black-Scholes formula. However $\sigma$ cannot be expressed as a function of $S_0, E, r, t, c$ and therefore a numerical method must be employed:

a. By trial and error. Begin with some value of $\sigma$ and compute $c$ using the Black-Scholes model. If the price of $c$ is too low (compare to the market price) increase $\sigma$ and iterate the procedure until the value of $c$ in the market is found. Note: the price of the call increases with volatility.

b. Use the method of Newton-Raphson to estimate $\sigma$. The method works as follows:

\[ c = S_0 \Phi(d_1) - \frac{E}{e^{rt}} \Phi(d_2) \Rightarrow f(\sigma) = S_0 \Phi(d_1) - \frac{E}{e^{rt}} \Phi(d_2) - c = 0. \]

\[ d_1 = \frac{\ln \left( \frac{S_0}{E} \right) + (r + \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \]

\[ d_2 = \frac{\ln \left( \frac{S_0}{E} \right) + (r - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} = d_1 - \sigma \sqrt{t} \]

To find $\sigma$ we begin with an initial value $\sigma_0$ and iterate as follows:

\[ \sigma_{i+1} = \sigma_i - \frac{f(\sigma_i)}{f'(\sigma_i)} \]

\[ i = 0 \]

\[ \sigma_1 = \sigma_0 - \frac{f(\sigma_0)}{f'(\sigma_0)} \]

\[ i = 1 \]

\[ \sigma_2 = \sigma_1 - \frac{f(\sigma_1)}{f'(\sigma_1)} \]

\[ \vdots \]

The procedure stops when the $|\sigma_{n+1} - \sigma_n|$ is small.

Note:
The derivative of $f(\sigma)$ above is

\[ f'(\sigma) = S_0 f(d_1) \times d_1' - \frac{E}{e^{rt}} f(d_2) \times d_2' \]

where $f(d_1)$ is the density of the standard normal distribution at $d_1$, i.e.

\[ f(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{d_1 - 0}{1} \right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_1^2} \]

Similarly,

\[ f(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{d_2 - 0}{1} \right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_2^2} \]
Example:
Suppose the value of a European call is \( C = 1.875 \) when \( s_0 = 21, E = 20, r = 0.1, t = 0.25 \). Use the method of Newton-Raphson to compute the implied volatility:

```r
#Inputs:
s0 <- 21
e <- 20
r <- 0.1
t <- 0.25
c <- 1.875

#Initial value of volatility:
sigma <- 0.10
sig <- rep(0,10)
sig[1] <- sigma

#Newton-Raphson method:
for(i in 2:100){
d1 <- (log(s0/E)+(r+sigma^2/2)*t)/(sigma*sqrt(t))
d2 <- d1-sigma*sqrt(t)
f <- s0*pnorm(d1)-E*exp(-r*t)*pnorm(d2)-c

#Derivative of d1 w.r.t. sigma:
d11 <- (sigma^2*t*sqrt(t)-(log(s0/E)+(r+sigma^2/2)*t)*sqrt(t))/(sigma^2*t)

#Derivative of d2 w.r.t. sigma:
d22 <- d11-sqrt(t)

#Derivative of f(sigma):
f1 <- s0*dnorm(d1)*d11-E*exp(-r*t)*dnorm(d2)*d22

#Update sigma:
sigma <- sigma - f/f1
sig[i] <- sigma
if(abs(sig[i]-sig[i-1]) < 0.00000001){sig<- sig[1:i]; break}
}
```

Here is the vector that contains the volatility at each step:

```r
> sig
[1] 0.1000000 0.3575822 0.2396918 0.2345343 0.2345129 0.2345129
```

The implied volatility is \( \sigma = 0.2345 \).

The graph shows the plot of the function \( f(\sigma) \) against \( \sigma \). The implied volatility is the value of \( \sigma \) such that \( f(\sigma) = 0 \).