Ito’s lemma, lognormal property of stock prices
Black Scholes Model


A. Ito’s lemma:
Ito’s lemma gives a derivative chain rule of random variables. Let $G$ be a function of $(S,t)$. Ito’s lemma states that $G$ follows the generalized Wiener process as follows:

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt + \frac{\partial G}{\partial S} \sigma S \epsilon \sqrt{dt}$$

(1)

B. The lognormal property of stock prices:
Let $G = \ln S$, and let us apply Ito’s lemma to this function (here $G$ is only function of $S$). Therefore

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \quad \frac{\partial G}{\partial t} = 0$$

And (1) will be:

$$dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma \epsilon \sqrt{dt}$$

Therefore, $G$ follows the generalized Wiener process with drift rate $\mu - \frac{\sigma^2}{2}$ and variance $\sigma^2$. As always, $\epsilon \sim N(0,1)$. Examine the change in $G$ from time $t$ (now) to time $T$: This can be expressed as: $\Delta G = \ln(S_T) - \ln(S_0)$. What is the distribution of this change? Reminder: $\Delta G$ follows the generalized Wiener process.

Example:
Let $S = $40, and $\mu = 0.16, \sigma = 0.20$ per year.

a. Find the distribution of $\ln S_T$ in 6 months.

b. Find $P(a < S_T < b) = 0.95$. 
Mean and variance of the stock price at time $T$:

Use moment generating functions to find $E(S_T)$ and $var(S_T)$. Reminder: The mgf of a normal random variable $X$ with mean $\mu$ and standard deviation $\sigma$ is

$$M_X(t) = E(e^{t^*X}) = e^{\mu t^* + \frac{1}{2} \sigma^2 t^*}$$

Here we use $t^*$ so that we don’t confuse it with $t$ which stands for time. In our case the random variable is $Y = \ln S_T$. Therefore the mgf of $Y$ is:

$$M_Y(t^*) = E(e^{t^*Y}) = E(e^{t^*\ln S_T}) = E(e^{\ln S_T t^*}) = E(S_T)^{t^*} \tag{2}$$

But earlier we found that:

$$Y = \ln S_T \sim N\left(\ln S + (\mu - \frac{\sigma^2}{2})(T - t), \sigma \sqrt{T - t}\right)$$

Therefore using the mgf of the normal random variable (2) will be:

$$M_Y(t^*) = E(S_T)^{t^*} = e^{(\ln S + (\mu - \frac{\sigma^2}{2})(T - t)) t^* + \frac{1}{2} \sigma^2(t - t) t^*^2}$$

Therefore when $t^* = 1$ we will get $E(S_T)$ and when $t^* = 2$ we will get $E(S_T^2)$.

$$E(S_T) = e^{\ln S} e^\mu(T - t) = S e^{\mu(T - t)}$$

and

$$E(S_T^2) = S^2 e^{2\mu(T - t) + \sigma^2(T - t)}$$

Now combining $E(S_T)$ and $E(S_T^2)$ we can find the variance of $S_T$:

$$var(S_T) = E(S_T^2) - (E(S_T))^2 = S^2 e^{2\mu(T - t)} \left( e^{\sigma^2 (T - t)} - 1 \right)$$

Example:

A stock has a current price $20$, and $\mu = 0.20, \sigma = 0.40$ per year. Find its expected price and variance in 1 year from now.

On the next page we see the calculations for estimating volatility $\sigma$. 
Estimate annual volatility for AAPL:

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### Data Table

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### Calculations

\[
\begin{align*}
\text{sum}(u) &= 0.06481782 \\
\text{sum}(u^2) &= 0.01203149
\end{align*}
\]
C. Black Scholes model:

A call option is a function of $S$ (stock price) and $t$ (time). Let $C$ be the price of the call option. Then from Ito’s lemma we have:

$$dC = \left( \frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} \sigma S \epsilon \sqrt{dt}$$  \hspace{1cm} (3)

Similar to the binomial option pricing model we want to create a riskless portfolio by

- Buying the call.
- Sell $n$ shares of the stock per call.

Then, the portfolio at time 0 is:

$$C - nS = \Pi$$

This portfolio will change from time $t$ to time $t + dt$ as follows:

$$dC - ndS = d\Pi$$  \hspace{1cm} (4)

But, $dc$ is given by (3) and also $S$ follows generalized Wiener process, that is:

$$dS = \mu S dt + \sigma S \epsilon \sqrt{dt}$$

Therefore (4) is:

$$d\Pi = \left( \frac{\partial C}{\partial S} \sigma S \epsilon \sqrt{dt} + \left( \frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - n \mu S \right) dt \right.$$  \hspace{1cm} (5)

or

$$d\Pi = \left( \frac{\partial C}{\partial S} \sigma \epsilon \sqrt{dt} + \left( \frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - n \mu S \right) dt \right.$$  \hspace{1cm} (6)

This will be a riskless portfolio if we eliminate the term involving $\epsilon$ (the only random component) from the above expression. So if we choose $n = \frac{\partial C}{\partial S}$ then

$$d\Pi = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial S^2} \right) dt$$  \hspace{1cm} (7)

One more step: Since this is a riskless portfolio it must earn the risk free rate during time $dt$

$$r \Pi dt = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial S^2} \right) dt$$

Also, $\Pi = C - nS$ and $n = \frac{\partial C}{\partial S}$. Putting all these together we get the Black-Scholes differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 + rS \frac{\partial C}{\partial S} - rC = 0$$

The solution of this differential equation gives the Black-Scholes option pricing formula:
The value $C$ of a European call option at time $t = 0$ is:

$$C = S_0 \Phi(d_1) - \frac{E}{e^{rt}} \Phi(d_2)$$

$$d_1 = \frac{ln \left( \frac{S_0}{E} \right) + (r + \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}}$$

$$d_2 = \frac{ln \left( \frac{S_0}{E} \right) + (r - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} = d_1 - \sigma \sqrt{t}$$

- $S_0$: Price of the stock at time $t = 0$
- $E$: Exercise price at expiration
- $r$: Continuously compounded risk-free interest
- $\sigma$: Annual standard deviation of the returns of the stock
- $t$: Time to expiration in years
- $\Phi(d_i)$: Cumulative probability at $d_i$ of the standard normal distribution $N(0,1)$, that is, $\Phi(d_i) = P(Z \leq d_i)$

Example:
Use the Black-Scholes option pricing formula to find the price of the European call if $S_0 = 30$, $E = 29$, days to expiration $t = 40$, annual standard deviation $\sigma = 0.30$, and continuously compounded risk-free interest rate $r = 0.05$. 