Multi-index model
Short sales allowed

From: “Simple Rules for Optimal Portfolio Selection: The Multi Group Case”

Stocks are grouped by industry. The multi-index model used here gives a diagonal form of the covariance matrix between stocks. The assumptions for this model is that stocks are linearly related to the group index (industry) and the industry is linearly related to the market index. Here is the model:

\[ R_i = \alpha_i + \beta_i I_j + \epsilon_i \]
\[ I_j = \gamma_j + b_j R_m + c_j \]

with,
\[ E(\epsilon_i \epsilon_k) = 0 \quad i = 1, \ldots, n, \quad k = 1, \ldots, n, \quad i \neq k \]
\[ E(c_j c_l) = 0 \quad j = 1, \ldots, p, \quad l = 1, \ldots, p, \quad j \neq l \]
\[ E(\epsilon_i c_j) = 0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, p \]

where,
\( R_i \) Return of stock \( i \)
\( I_j \) Return of index \( j \)
\( R_m \) Return of the market
\( \alpha_i, \beta_i \) Parameters associated with stock \( i \)
\( \gamma_j, b_j \) Parameters associated with index \( j \)
\( \epsilon_i \) Error term with mean zero and variance \( \sigma_{\epsilon_i}^2 \)
\( c_j \) Error term with mean zero and variance \( \sigma_{c_j}^2 \)

Using these assumptions we get the following.

Variance of the return of stock \( i \):
\[ \sigma_i^2 = \beta_i^2 \sigma_j^2 + \sigma_{\epsilon_i}^2 \]
But \[ \sigma_j^2 = b_j^2 \sigma_m^2 + \sigma_{c_j}^2 \]
Therefore, \[ \sigma_i^2 = \beta_i^2 (b_j^2 \sigma_m^2 + \sigma_{c_j}^2) + \sigma_{\epsilon_i}^2 \] (1)

Covariance between stocks \( i \) and \( k \) in the same group (industry):
\[ \sigma_{ik} = \beta_i \beta_k (b_j^2 \sigma_m^2 + \sigma_{c_j}^2) \] (2)

Covariance between stocks \( i \) and \( k \) in different industries:
\[ \sigma_{ik} = \beta_i \beta_k b_l \sigma_m^2 \] (3)
Assume two stocks and two industries (two per industry). The solution as always is given by the following system of equations:

\[\bar{R}_1 - R_f = z_1 \sigma_{c1}^2 + z_2 \sigma_{12} + z_3 \sigma_{13} + z_4 \sigma_{14}\]  \hspace{1cm} (4)

\[\bar{R}_2 - R_f = z_1 \sigma_{21} + z_2 \sigma_{22}^2 + z_3 \sigma_{23} + z_4 \sigma_{24}\]  \hspace{1cm} (5)

\[\bar{R}_3 - R_f = z_1 \sigma_{31} + z_2 \sigma_{32} + z_3 \sigma_{33}^2 + z_4 \sigma_{34}\]  \hspace{1cm} (6)

\[\bar{R}_4 - R_f = z_1 \sigma_{41} + z_2 \sigma_{42} + z_3 \sigma_{43} + z_4 \sigma_{44}\]  \hspace{1cm} (7)

Let’s examine equation (4) and see how it can be written using equations (1), (2), and (3).

\[\bar{R}_1 - R_f = z_1 \left( \beta_1^2 [b_1^2 \sigma_m^2 + \sigma_{c1}^2] + \sigma_{c1}^2 \right) + z_2 \left( \beta_1 \beta_2 [b_1^2 \sigma_m^2 + \sigma_{c1}^2] \right) + z_3 \left( \beta_1 \beta_2 b_1 b_2 \sigma_m^2 \right) + z_4 \left( \beta_1 \beta_2 b_1 b_2 \sigma_m^2 \right)\]

Rearrange

\[\bar{R}_1 - R_f = z_1 \sigma_{c1}^2 + \beta_1 \left[ z_1 \beta_1 b_1^2 \sigma_m^2 + z_2 \beta_2 b_1^2 \sigma_m^2 + z_1 \beta_1 \sigma_{c1}^2 + z_2 \beta_2 \sigma_{c1}^2 \right]
+ \beta_1 \left[ z_3 \beta_3 b_1 b_2 \sigma_m^2 + z_4 \beta_4 b_1 b_2 \sigma_m^2 \right]
\]

Or

\[\bar{R}_1 - R_f = z_1 \sigma_{c1}^2 + \beta_1 \left[ b_1^2 \sigma_m^2 (z_1 \beta_1 + z_2 \beta_2) + \sigma_{c1}^2 (z_1 \beta_1 + z_2 \beta_2) \right] + \beta_1 \left[ b_1 b_2 \sigma_m^2 (z_3 \beta_3 + z_4 \beta_4) \right]\]

If we let \( \Phi_1 = z_1 \beta_1 + z_2 \beta_2 \) and \( \Phi_2 = z_3 \beta_3 + z_4 \beta_4 \) we get:

\[\bar{R}_1 - R_f = z_1 \sigma_{c1}^2 + \beta_1 \left[ (\sigma_{c1}^2 + b_1^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2 \right]
\]

Now solve for \( z_1 \):

\[z_1 = \frac{\beta_1}{\sigma_{c1}^2} \left[ \frac{\bar{R}_1 - R_f}{\beta_1} - \left[ (\sigma_{c1}^2 + b_1^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2 \right] \right]\]  \hspace{1cm} (8)

Similarly, stock 2 will give the following expression:

\[z_2 = \frac{\beta_2}{\sigma_{c2}^2} \left[ \frac{\bar{R}_2 - R_f}{\beta_2} - \left[ (\sigma_{c2}^2 + b_2^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2 \right] \right]\]  \hspace{1cm} (9)

These expressions look similar to the single index model solution (see class notes). But now the \( C^* \) cut-off point is a more complicated expression and of course it is the same for stocks in the same industry. In our example the \( C^* \) for industry 1 is equal to:

\[C^*_1 = (\sigma_{c1}^2 + b_1^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2\]

If we know \( C^*_1 \) it will be easy to compute the \( z_i's \) and from there the \( x_i's \). In order to find \( C^*_1 \) we need to find \( \Phi_1 \) and \( \Phi_2 \).

Multiply (8) by \( \beta_1 \) and (9) by \( \beta_2 \):

\[z_1 \beta_1 = \frac{\beta_1^2}{\sigma_{c1}^2} \left[ \frac{\bar{R}_1 - R_f}{\beta_1} - \left[ (\sigma_{c1}^2 + b_1^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2 \right] \right]\]  \hspace{1cm} (10)

and

\[z_2 \beta_2 = \frac{\beta_2^2}{\sigma_{c2}^2} \left[ \frac{\bar{R}_2 - R_f}{\beta_2} - \left[ (\sigma_{c2}^2 + b_2^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2 \right] \right]\]  \hspace{1cm} (11)

To produce \( \Phi_1 \) on the left hand side add (10) and (11):
\[
\sum_{i=1}^{2} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{i}^2} = \Phi_1 + \frac{\beta_1^2}{\sigma_{c1}^2} \left[ \sigma_{c1}^2 + b_1^2 \sigma_m^2 \right] \Phi_1 + \frac{\beta_2^2}{\sigma_{c2}^2} b_1 b_2 \sigma_m^2 \Phi_2 \\
+ \frac{\beta_2^2}{\sigma_{c2}^2} \left[ \sigma_{c2}^2 + b_1^2 \sigma_m^2 \right] \Phi_1 + \frac{\beta_2^2}{\sigma_{c2}^2} b_1 b_2 \sigma_m^2 \Phi_2
\]

Or

\[
\sum_{i=1}^{2} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{i}^2} = \Phi_1 \left[ 1 + \frac{\beta_1^2}{\sigma_{c1}^2} \left[ \sigma_{c1}^2 + b_1^2 \sigma_m^2 \right] + \frac{\beta_2^2}{\sigma_{c2}^2} \left[ \sigma_{c2}^2 + b_1^2 \sigma_m^2 \right] \right] \\
+ \Phi_2 \left[ \frac{\beta_1^2}{\sigma_{c1}^2} b_1 b_2 \sigma_m^2 + \frac{\beta_2^2}{\sigma_{c2}^2} b_1 b_2 \sigma_m^2 \right]
\]

(12)

Similarly, for stocks 3 and 4 that belong in industry 2 we get:

\[
\sum_{i=3}^{4} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{i}^2} = \Phi_1 \left[ \frac{\beta_3^2}{\sigma_{c3}^2} b_1 b_2 \sigma_m^2 + \frac{\beta_4^2}{\sigma_{c4}^2} b_1 b_2 \sigma_m^2 \right] \\
+ \Phi_2 \left[ 1 + \frac{\beta_3^2}{\sigma_{c3}^2} \left[ \sigma_{c3}^2 + b_1^2 \sigma_m^2 \right] + \frac{\beta_4^2}{\sigma_{c4}^2} \left[ \sigma_{c4}^2 + b_1^2 \sigma_m^2 \right] \right]
\]

(13)

The system above can be written in vector and matrix form as \( M \Phi = R \) and therefore: \( \Phi = M^{-1} R \). The dimensions of the matrix \( M \) in our example are 2 \( \times \) 2 (two industries).
Multi-index system

\[
\Phi = \begin{pmatrix}
\Phi_1 \\
\Phi_2 
\end{pmatrix} = \begin{pmatrix}
1 + \frac{\beta_1^2}{\sigma_{c_1}^2} (\sigma_{c_1}^2 + b_1^2 \sigma_m^2) + \frac{\beta_2^2}{\sigma_{c_2}^2} (\sigma_{c_2}^2 + b_1^2 \sigma_m^2) & \left[ \frac{\beta_1 \beta_2 b_2}{\sigma_{c_1}^2} + \frac{\beta_1 b_2}{\sigma_{c_2}^2} \right] \sigma_m^2 \\
\left[ \frac{\beta_2 \beta_1 b_2}{\sigma_{c_2}^2} + \frac{\beta_2 b_1}{\sigma_{c_1}^2} \right] \sigma_m^2 & 1 + \frac{\beta_2^2}{\sigma_{c_2}^2} (\sigma_{c_2}^2 + b_2^2 \sigma_m^2) + \frac{\beta_2^2}{\sigma_{c_4}^2} (\sigma_{c_4}^2 + b_2^2 \sigma_m^2)
\end{pmatrix}^{-1} \begin{pmatrix}
\sum_{i=1}^{2} \left( \bar{R}_i - R_f \right) \beta_i \\
\sum_{i=3}^{4} \left( \bar{R}_i - R_f \right) \beta_i
\end{pmatrix}
\]

Once \( \Phi_1 \) and \( \Phi_2 \) are obtained we can compute the \( z_i \)’s (see equations (8) and (9)).

\[
z_1 = \frac{\beta_1}{\sigma_{c_1}^2} \left[ \frac{\bar{R}_1 - R_f}{\beta_1} - \left( (\sigma_{c_1}^2 + b_1^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2 \right) \right]
\]

\[
z_2 = \frac{\beta_2}{\sigma_{c_2}^2} \left[ \frac{\bar{R}_2 - R_f}{\beta_2} - \left( (\sigma_{c_2}^2 + b_1^2 \sigma_m^2) \Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2 \right) \right]
\]

\[
z_3 = \frac{\beta_3}{\sigma_{c_3}^2} \left[ \frac{\bar{R}_3 - R_f}{\beta_3} - \left( (\sigma_{c_3}^2 + b_2^2 \sigma_m^2) \Phi_2 + b_1 b_2 \sigma_m^2 \Phi_1 \right) \right]
\]

\[
z_4 = \frac{\beta_4}{\sigma_{c_4}^2} \left[ \frac{\bar{R}_4 - R_f}{\beta_4} - \left( (\sigma_{c_4}^2 + b_2^2 \sigma_m^2) \Phi_2 + b_1 b_2 \sigma_m^2 \Phi_1 \right) \right]
\]

Write the system in vector and matrix form when there are three industries with three stocks in each industry.