Problem 1 For a continuous variable $X \sim f(x)$. For each part, it is not sufficient to simply use the properties of expectation and variance.

(1) Prove $E[aX + b] = aE[X] + b$.

Proof.

\[
E[aX + b] = \int_{-\infty}^{\infty} (aX + b) f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} aX f(x) \, dx + \int_{-\infty}^{\infty} b f(x) \, dx
\]

\[
= a \int_{-\infty}^{\infty} X f(x) \, dx + b \int_{-\infty}^{\infty} f(x) \, dx
\]

\[
= a \mu + b \times \int_{-\infty}^{\infty} f(x) \, dx
\]

\[
= a \mu + b \times 1
\]

\[
= aE[X] + b
\]

(2) Prove $\text{Var}(aX + b) = a^2 \text{Var}(X)$. Let $Z = aX + b$.

Proof.

\[
\text{Var}(Z) = \int_{-\infty}^{\infty} (z - E(Z))^2 f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} (ax + b - [a\mu + b])^2 f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} (ax - a\mu)^2 f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} a^2 (x - \mu)^2 f(x) \, dx
\]

\[
= a^2 \int_{-\infty}^{\infty} (x - E[X])^2 f(x) \, dx
\]

\[
= a^2 \text{Var}(X)
\]
(3) Prove $\text{Var}(X) = E[X^2] - E[X]^2$.

Proof.

\[
\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \\
= \int_{-\infty}^{\infty} (x^2 + \mu^2 - 2x\mu) f(x) \, dx \\
= \int_{-\infty}^{\infty} x^2 f(x) \, dx + \mu^2 \int_{-\infty}^{\infty} f(x) \, dx - 2\mu \int_{-\infty}^{\infty} x f(x) \, dx \\
= E[X^2] + \mu^2 - 2\mu^2 \\
= E[X^2] - \mu^2 \\
= E[X^2] - E[X]^2
\]

(4) Let $\mu = E[X], \sigma^2 = \text{Var}(X)$ and $Z = \frac{X-\mu}{\sigma}$. Calculate $E[Z]$ and $\text{Var}(Z)$.

Let $Z = \frac{X-\mu}{\sigma}$.

It is expected that you compute $E(Z)$ and $\text{Var}(Z)$ as follows

\[
E[Z] = \int_{-\infty}^{\infty} z f(z) \, dz \\
= \int_{-\infty}^{\infty} \left( \frac{x - \mu}{\sigma} \right) f(x) \, dx \\
= \frac{1}{\sigma} \int_{-\infty}^{\infty} (x - \mu) f(x) \, dx \\
= \frac{1}{\sigma} \left( \int_{-\infty}^{\infty} x f(x) \, dx - \mu \int_{-\infty}^{\infty} f(x) \, dx \right) \\
= \frac{1}{\sigma} (E(X) - \mu) \\
= \frac{\mu - \mu}{\sigma} \\
= 0
\]
But it is also acceptable to use what we just proved.

\[
E[Z] = E \left( \frac{X - \mu}{\sigma} \right) \\
= \frac{1}{\sigma} E(X - \mu) \\
= \frac{1}{\sigma} [E(X) - \mu] \\
= \frac{1}{\sigma} (\mu - \mu) \\
= 0
\]

Similarly,

\[
\text{Var}(Z) = \text{Var} \left( \frac{X - \mu}{\sigma} \right) \\
= \frac{1}{\sigma^2} \text{Var}(X) \\
= \frac{1}{\sigma^2} \cdot \sigma^2 \\
= 1
\]

**Problem 2**

*For* \( U \sim \text{Uniform}[0, 1] \), *calculate* \( E[U] \), \( E[U^2] \), \( \text{Var}(U) \), *and* \( F(u) = P(U \leq u) \).

By definition, the PDF for the uniform distribution is \( f(x) = \frac{1}{b-a} \). In this problem, \( b = 1 \) and \( a = 0 \) so \( f(x) = 1 \). Then,

\[
E(U) = \int_0^1 uf(u) \, du \\
= \int_0^1 u \, du \\
= \frac{1}{2} u^2 \bigg|_0^1 \\
= \frac{1}{2}
\]

In general, for \( U \sim \text{Uniform}[a, b] \),

\[
E(U) = \frac{b - a}{2}
\]
\[ E(U^2) = \int_0^1 u^2 f(u) \, du \]
\[ = \int_0^1 u^2 \, du \]
\[ = \frac{1}{3} \left| u^3 \right|_0^1 \]
\[ = \frac{1}{3} \]

Then

\[ \text{Var}(U) = E(U^2) - E(U)^2 \]
\[ = \frac{1}{3} - \left( \frac{1}{2} \right)^2 \]
\[ = \frac{1}{3} - \frac{1}{4} \]
\[ = \frac{1}{12} \]

and in general, for \( U \sim \text{Uniform}[a, b] \),

\[ \text{Var}(U) = \frac{b - a}{12} \]

**Problem 3**

For \( T \sim \text{Exp}(\lambda) \),

1. Calculate \( F(t) = P(T \leq t) \). For \( a > b > 0 \), find \( P(T > a | T > b) \).

By definition
\[ F(t) = P(T \leq t) \]
\[ = \int_0^t \lambda e^{-\lambda t} \, dt \]
\[ = \lambda \left[ -\frac{1}{\lambda} e^{-\lambda t} \right]_0^t \]
\[ = -\left( e^{-\lambda t} - 1 \right) \]
\[ = 1 - e^{-\lambda t} \]

Thus, we can also see that \( P(T > t) = e^{-\lambda t} \).
(2) **Calculate** \( E[T] \) **and** \( \text{Var}(T) \).

By definition

\[
E(T) = \int_0^\infty t \lambda e^{-\lambda t} \, dt
\]

\[
= \lambda \int_0^\infty te^{-\lambda t} \, dt
\]

Use integration by parts, \( u = t, dv = e^{-\lambda t} \) and the IBP formula \( uv - \int v \, du \).

\[
= \lambda \left[ t \left( -\frac{1}{\lambda} e^{-\lambda t} \right) - \int -\frac{1}{\lambda} e^{-\lambda t} \, dt \right]
\]

\[
= \lambda \left[ t \left( -\frac{1}{\lambda} e^{-\lambda t} \right) + \frac{1}{\lambda} \int e^{-\lambda t} \, dt \right]
\]

\[
= \lambda \left[ t \left( -\frac{1}{\lambda} e^{-\lambda t} \right) + \frac{1}{\lambda} \left( -\frac{1}{\lambda} \right) e^{-\lambda t} \right]_0^\infty
\]

\[
= \lambda \left[ t \left( -\frac{1}{\lambda} e^{-\lambda t} \right) - \frac{1}{\lambda^2} e^{-\lambda t} \right]_0^\infty
\]

\[
= -te^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t} \bigg|_0^\infty
\]

\[
= \left[ -e^{-\lambda t} \left( t + \frac{1}{\lambda} \right) \right]_0^\infty
\]

Note that to evaluate at \( \infty \), we take the limit and that

\[
\lim_{t \to \infty} \left( -e^{-\lambda t} \right) \left( t + \frac{1}{\lambda} \right) = \infty \cdot 0
\]

which is an indeterminate form, but we can do

\[
\lim_{t \to \infty} - \left( t + \frac{1}{\lambda} \right) = \frac{-\infty}{\infty}
\]

another indeterminate form, but now we can use L’Hopital’s Rule yielding

\[
\lim_{t \to \infty} \frac{1}{\lambda e^{\lambda t}} = 0
\]

Then,

\[
\left[ -e^{-\lambda t} \left( t + \frac{1}{\lambda} \right) \right]_0^\infty = 0 - \left[ -\frac{1}{\lambda} \right] = \frac{1}{\lambda}
\]
Now we compute \( \text{Var}(T) \). Recall that \( \text{Var}(T) = E(T^2) - E(T)^2 \). We find \( E(T^2) \).

\[
E(T^2) = \int_0^\infty t^2 \lambda e^{-\lambda t} \, dt
\]

\[
= \lambda \int_0^\infty t^2 e^{-\lambda t} \, dt
\]

By integration by parts, \( u = t^2, dv = e^{-\lambda t} \) and the IBP formula \( uv - \int v \, du \).

\[
= \lambda \left[ \int_0^\infty \left( -\frac{1}{\lambda} e^{-\lambda t} \right) - \int \frac{1}{\lambda} \cdot 2t \cdot e^{-\lambda t} \, dt \right]_0^\infty
\]

\[
= \lambda \left[ -\frac{t^2}{\lambda} e^{-\lambda t} + \frac{2}{\lambda} \int t e^{-\lambda t} \, dt \right]_0^\infty
\]

\[
= -t^2 e^{-\lambda t} + 2 \int t e^{-\lambda t} \, dt
\]

By integration by parts, \( u = t, dv = e^{-\lambda t} \) and the IBP formula \( uv - \int v \, du \).

\[
= -t^2 e^{-\lambda t} + 2 \left\{ \left( \frac{1}{\lambda} e^{-\lambda t} \right) - \int \frac{1}{\lambda} e^{-\lambda t} \, dt \right\}_0^\infty
\]

\[
= -t^2 e^{-\lambda t} + 2 \left\{ -\frac{t}{\lambda} e^{-\lambda t} + \frac{1}{\lambda} \int e^{-\lambda t} \, dt \right\}_0^\infty
\]

\[
= -t^2 e^{-\lambda t} + 2 \left\{ -\frac{t}{\lambda} e^{-\lambda t} + \frac{1}{\lambda} \left( -\frac{1}{\lambda} \right) e^{-\lambda t} \right\}_0^\infty
\]

\[
= -t^2 e^{-\lambda t} + 2 \left\{ -\frac{t}{\lambda} e^{-\lambda t} - \frac{1}{\lambda^2} e^{-\lambda t} \right\}_0^\infty
\]

\[
= -t^2 e^{-\lambda t} - \frac{2t}{\lambda} e^{-\lambda t} - \frac{2}{\lambda^2} e^{-\lambda t}
\]

\[
= -e^{-\lambda t} \left( t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \right)
\]

\[
\lim_{t \to \infty} -e^{-\lambda t} \left( t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \right) = 0 \cdot \infty
\]

which is indeterminate and can be arranged to a friendly indeterminate form

\[
\lim_{t \to \infty} -\frac{t^2}{\lambda} + \frac{2t}{\lambda} + \frac{2}{\lambda^2} e^{\lambda t} = \infty \cdot \frac{\infty}{\infty}
\]

L’Hopital’s Rule then yields

\[
\lim_{t \to \infty} -\frac{2t + \frac{2}{\lambda}}{\lambda e^{\lambda t}} = \infty
\]

So we use L’Hopital’s Rule again, which yields

\[
\lim_{t \to \infty} -\frac{2t}{\lambda^2 e^{\lambda t}} = 0
\]
So we get

\[ E(T^2) = -e^{-\lambda t} \left( t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \right) \bigg|_0^\infty = \left[ 0 - \left( -\frac{2}{\lambda^2} \right) \right] = \frac{2}{\lambda^2} \]

\[ \text{Var}(T) = E(T^2) - (E(T))^2 \]
\[ = \frac{2}{\lambda^2} - \left( \frac{1}{\lambda} \right)^2 \]
\[ = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \]

(3) If \( U \sim \text{Uniform}[0,1] \), let \( X = -\frac{\log U}{\lambda} \). Calculate \( F(x) = P(X \leq x) \), and \( f(x) = F(x) \). What is the distribution of \( X \)?

To find the distribution of \( X \), we need to find \( f_X(x) \). We do this by starting with the CDF of \( x \) and differentiating it with respect to \( x \).

\[ F_X(x) = P(X \leq x) \]
\[ = P \left( -\frac{\log U}{\lambda} \leq x \right) \]
\[ = P \left( \log U \geq -\lambda x \right) \]
\[ = P \left( U \geq e^{-\lambda x} \right) \]
\[ = 1 - P \left( U \leq e^{-\lambda x} \right) \]

But notice that \( P \left( U \leq e^{-\lambda x} \right) \) is the CDF for \( U \), \( F_U(u) \).

\[ = 1 - F_U \left( e^{-\lambda x} \right) \]

Recall from an earlier problem that the CDF for the uniform distribution is \( F_U(u) = u \), so replace \( u \) with \( e^{-\lambda x} \).

\[ = 1 - e^{-\lambda x} \]

And notice that the above is the CDF for the exponential distribution.

To get the distribution of \( X \), differentiation w/r/t \( x \).

\[ f(x) = \lambda e^{\lambda x} \]

which is the PDF \( f(x) \) for the exponential distribution with parameter \( \lambda \), thus \( X \sim \text{Exp}(\lambda) \).
Let $Y = aT$ where $a > 0$. What is the density function of $Y$?

Again, to get $f(y)$, we start with the CDF of $Y$, $F_Y$ and then differentiate it with respect to $y$.

$$4F_Y(y) = P(Y \leq y) = P(aT \leq y) = P\left(T \leq \frac{y}{a}\right)$$

which is the CDF for $T$ w/r/t $y$, $F_T(y)$.  

$$= F_T\left(\frac{y}{a}\right)$$

Now differentiate with respect to $y$.

$$= f_T\left(\frac{y}{a}\right) \cdot \frac{dy}{dy} a$$

$$= \frac{1}{a} f_T\left(\frac{y}{a}\right)$$

where $f_T(t) = \lambda e^{-\lambda t}$

$$= \frac{1}{a} \cdot \lambda e^{-\frac{ty}{a}}$$

$$= \frac{\lambda}{a} e^{-\left(\frac{y}{a}\right)}$$

Thus $Y \sim \text{Exp}\left(\frac{\lambda}{a}\right)$.

**Problem 4**

Suppose $Z \sim N(0, 1)$. The density of $Z$ is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

(1) Calculate $E[Z]$ and $\text{Var}[Z]$.

By definition,

$$E(Z) = \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

If we pull out the annoying constant, we get

$$E(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}} dz$$
Trick: Note that the function \(ze^{\frac{z^2}{2}}\) is an odd function. Recall from Math 31B that if \(f(x)\) is odd, then

\[
\int_{-a}^{a} f(x) \, dx = 0
\]

Thus, \(E(Z) = 0\).

Recall that

\[
\operatorname{Var}(Z) = \int_{-\infty}^{\infty} (z - E(Z))^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \cdot e^{-\frac{z^2}{2}} \, dz
\]

We integrate by parts. Let \(u = z, \, dv = ze^{\frac{-z^2}{2}}\). Then,

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \cdot e^{-\frac{z^2}{2}} \, dz = \frac{1}{\sqrt{2\pi}} \left\{ \left[-z e^{-\frac{z^2}{2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz \right\}
\]

But note that since \(ze^{\frac{-z^2}{2}}\) is odd, \([-z e^{-\frac{z^2}{2}}]_{-\infty}^{\infty} = 0\).

So we are left with

\[
\operatorname{Var}(Z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz
\]

But note that the integrand is the standard normal distribution! This means that when integrated over its domain (infinite), the integral is 1!

So \(\operatorname{Var}(Z) = 1\).

(2) Let \(X = \mu + \sigma Z\). What are \(E[X]\) and \(\operatorname{Var}(X)\)? What is the density function of \(X\)? Please! One at a time! No pushing... Let’s start with \(E[X]\). Since \(X = \mu + \sigma Z\),

\[
E(X) = E(\mu + \sigma Z) \\
= E(\mu) + E(\sigma Z) \\
= \mu + \sigma E(Z) \\
= \mu
\]

And,

\[
\operatorname{Var}(X) = \operatorname{Var}(\mu + \sigma Z) \\
= \sigma^2 \operatorname{Var}(Z) \\
= \sigma^2
\]
Now the hard part. We want to find the density function of $X$. There is a theorem that you will learn in 100B that is related to this problem. That theorem states that if $Z$ is a standard normal random variable ($Z \sim N(0, 1)$), then $X = \mu + \sigma Z$ is also a normal random variable. We just proved that the mean is $\mu$ and the variance is $\sigma$. What we have not proven is that the distribution of $X$ is normal... yet.

We start with the CDF of and then differentiate it to get the density function $f(x)$.

In my notation below, $F_X$ denotes that we are finding the CDF of $X$.

\[
F_X(x) = P(X \leq x) = P(\mu + \sigma Z \leq x) = P \left( Z \leq \frac{x - \mu}{\sigma} \right) = \int_{-\infty}^{\frac{x - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dt = F_Z \left( \frac{x - \mu}{\sigma} \right)
\]

Now we have that $F_X(x) = F_Z \left( \frac{x - \mu}{\sigma} \right)$ where $F_Z$ is the cumulative distribution function of the distribution of $Z$.

Recall that we want to differentiate $F_X(x)$ so that we get $f_X(x)$ (which is the density function), but since $F_X(x) = F_Z \left( \frac{x - \mu}{\sigma} \right)$, differentiating $F_X$ is the same as differentiating $F_Z$. Is everybody with me?

So we proceed as follows.

\[
\frac{d}{dx} F_X(x) = \frac{d}{dx} F_Z \left( \frac{x - \mu}{\sigma} \right) = f_Z \left( \frac{x - \mu}{\sigma} \right) \cdot \frac{1}{\sigma} = \frac{1}{\sigma} \cdot f_Z \left( \frac{x - \mu}{\sigma} \right)
\]

**Recap:** Before concluding, let’s review what we have calculated so far...

\[
\frac{d}{dx} F_X(x) = f_X(x) = \frac{1}{\sigma} \cdot f_Z \left( \frac{x - \mu}{\sigma} \right)
\]

But recall that

\[
f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}
\]
So

\[ f_Z \left( \frac{x - \mu}{\sigma} \right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \]

Thus,

\[ f_X(x) = \frac{1}{\sigma} \cdot f_Z \left( \frac{x - \mu}{\sigma} \right) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \]

Cool, eh?

**Problem 5**

*Suppose we flip a fair coin 1000 times independently. Let \( X \) be the number of heads. Answer the following questions using normal approximation.*

Note that we would usually solve this problem using the binomial. The computation would be cumbersome, but more importantly, since \( p = \frac{1}{2} \) is not too close to 0 or 1, and \( n \) is large, and since \( np > 10, n(1 - p) > 10 \), we use the normal approximation to the binomial. Since the binomial distribution is discrete and the normal distribution is continuous, we must use a correction.

Using the normal approximation, we have that \( \mu = np = \frac{1000}{2} = 500 \) and \( \sigma^2 = np(1 - p) = 500 \cdot \frac{1}{2} = 250 \).

1. **What is the probability that 480 ≤ \( X \) ≤ 520?**

\[
P(480 \leq X \leq 520) \approx P(479.5 \leq X \leq 520.5)
= P \left( \frac{479.5 - 500}{\sqrt{250}} \leq Z \leq \frac{520.5 - 500}{\sqrt{250}} \right)
= P(-1.3 \leq Z \leq 1.3)
= P(Z \leq 1.3) - P(Z \leq -1.3)
\approx 0.794
\]

Note that the range around the mean is symmetric so it is also true that

\[ P(Z \leq 1.3) - P(Z \leq -1.3) = 1 - 2P(Z \leq -1.3) = 0.794 \]

2. **What is the probability that \( X > 530 \)?**

\[
P(X > 530) = 1 - P(X \leq 530)
\approx 1 - P(X \leq 529.5)
= 1 - P \left( Z \leq \frac{529.5 - 500}{\sqrt{250}} \right)
= 1 - P(Z \leq 1.87)
\approx 0.031
\]
Problem 6

Suppose among the population of voters, $\frac{1}{3}$ of the people support a candidate. If we sample 1000 people from the population, and let $X$ be the number of supporters of this candidate among these 1000 people. Let $\hat{p} = \frac{X}{n}$ be the sample proportion. Answer the following questions using normal approximation.

Note that by the Central Limit Theorem, $\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right) = 0.0002$.

(1) What is the probability that $\hat{p} > .35$

\[
P(\hat{p} > 0.35) = P\left(\hat{p} > \frac{0.35 - 0.33}{\sqrt{0.0002}}\right) = 1 - P(Z \leq 1.41) = 0.079
\]

(2) What is the probability that $\hat{p} < .3$?

\[
P(\hat{p} < 0.3) = P\left(Z < \frac{0.3 - 0.33}{\sqrt{0.0002}}\right) = P(Z < -2.12) = 0.017
\]

Problem 7

Consider the following joint probability mass function $p(x, y)$ of the discrete random variables $(X, Y)$:

\[
x/y | 1 | 2 | 3 \\
1 | 0.1 | 0.1 | 0.1 \\
2 | 0.2 | 0.1 | 0.2 \\
3 | 0.1 | 0.05 | 0.05 \\
\]

(1) Calculate $p_X(x)$ for $x = 1, 2, 3$. Calculate $p_Y(y)$ for $y = 1, 2, 3$.

Note that $p_X(x)$ is the marginal of $X$ and is just the row sums, thus,

\[
p_X(1) = 0.1 + 0.1 + 0.1 = 0.3 \\
p_X(2) = 0.2 + 0.1 + 0.2 = 0.5 \\
p_X(3) = 0.1 + 0.05 + 0.05 = 0.2
\]

Note that $p_Y(y)$ is the marginal of $Y$ and is just the column sums, thus,

\[
p_Y(1) = 0.1 + 0.2 + 0.1 = 0.4 \\
p_Y(2) = 0.1 + 0.1 + 0.05 = 0.25 \\
p_Y(3) = 0.1 + 0.2 + 0.05 = 0.35
\]
(2) Calculate $P(X = x | Y = y)$ and calculate $P(Y = y | X = x)$ for all pairs of $(x, y)$.

By Bayes’ Rule,

$$P(X = x | Y = y) = \frac{p(x, y)}{P(Y = y)}$$
$$P(Y = y | X = x) = \frac{p(x, y)}{P(X = x)}$$

Thus, $P(X = x | Y = y)$:

<table>
<thead>
<tr>
<th>$x/y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.4</td>
<td>0.286</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.4</td>
<td>0.57</td>
</tr>
<tr>
<td>3</td>
<td>0.25</td>
<td>0.2</td>
<td>0.143</td>
</tr>
</tbody>
</table>

Thus, $P(Y = y | X = x)$:

<table>
<thead>
<tr>
<th>$x/y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.33</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

(3) Calculate $E(X)$ and $E(Y)$. Calculate $\text{Var}(X)$ and $\text{Var}(Y)$.

$$E(X) = 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3)$$
$$= 0.3 + 2 \cdot 0.5 + 3 \cdot 0.2$$
$$= 1.9$$

$$E(Y) = 1 \cdot P(Y = 1) + 2 \cdot P(Y = 2) + 3 \cdot P(Y = 3)$$
$$= 0.4 + 2 \cdot 0.25 + 3 \cdot 0.35$$
$$= 1.95$$

$$\text{Var}(X) = (1 - 1.9)^2 \cdot 0.3 + (2 - 1.9)^2 \cdot 0.5 + (3 - 1.9)^2 \cdot 0.2$$
$$= 0.49$$

$$\text{Var}(Y) = (1 - 1.95)^2 \cdot 0.4 + (2 - 1.95)^2 \cdot 0.25 + (3 - 1.95)^2 \cdot 0.35$$
$$= 0.7475$$
(4) *Calculate* $E(\text{XY})$. *Calculate* Cov$(X,Y)$. *Calculate* Corr$(X,Y)$.

Note that $E(\text{XY}) = E(X)E(Y)$ if and only if $X$ and $Y$ are independent.

\[
E(\text{XY}) = \sum_x \sum_y p(x, y)
\]

\[
= 1 \cdot 1 \cdot p(1, 1) + 1 \cdot 2 \cdot p(1, 2) + 1 \cdot 3 \cdot p(1, 3)
\]

\[
+ 2 \cdot 1 \cdot p(2, 1) + 2 \cdot 2 \cdot p(2, 2) + 2 \cdot 3 \cdot p(2, 3)
\]

\[
+ 3 \cdot 1 \cdot p(3, 1) + 3 \cdot 2 \cdot p(3, 2) + 3 \cdot 3 \cdot p(3, 3)
\]

\[
= 0.1 + 2 \cdot 0.1 + 3 \cdot 0.1 + 2 \cdot 0.2 + 4 \cdot 0.1 + 6 \cdot 0.2
\]

\[
+ 3 \cdot 0.1 + 6 \cdot 0.05 + 9 \cdot 0.05
\]

\[
= 3.65
\]

By definition, Cov$(X,Y) = E(\text{XY}) - E(X)E(Y)$.

\[
\text{Cov}(X, Y) = E(\text{XY}) - E(X)E(Y)
\]

\[
= 3.65 - 1.9 \cdot 1.95
\]

\[
= -0.055
\]

Since the covariance is not 0, $X$ and $Y$ are dependent. They have a negative relationship.

Since the covariance can be anything, we normalize to get a value between -1 (perfect negative correlation), and +1 (perfect positive correlation) where 0 means no correlation.

\[
\text{Corr}(X, Y) = \frac{\text{Cov}(X,Y)}{\sigma_x \sigma_y}
\]

\[
= \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}}
\]

\[
= \frac{-0.055}{\sqrt{0.49} \sqrt{0.7475}}
\]

\[
= -0.091
\]

$X$ and $Y$ have a weakly negative correlation.